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POINTWISE CONVERGENCE OF THE ITERATES OF A HARRIS-RECURRENT MARKOV OPERATOR

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ABSTRACT

Let P be a Markov operator recurrent in the sense of Harris, with σ-finite invariant measure μ. (1) If μ is finite and P aperiodic, then for $f \in L_1(\mu)$, $P^n f \to \int f d\mu$ a.e. (2) If μ is infinite, $P^n f \to 0$ a.e. for every $f \in L_p(\mu)$, $1 \leqq p < \infty$.

Let $P(x, A)$ be a transition probability on the measurable space (X, Σ), and denote also by P the operator on $B(X, \Sigma)$ defined by $Pf(x) = \int f(y)P(x, dy)$. P is *Harris-recurrent* if for a σ-finite measure m we have $m(A) > 0 \Rightarrow \sum_{n=0}^{\infty} P^n 1_A(x) = \infty$ for *every* x. It is well-known (see, for example, [2], [8]) that if P is Harris-recurrent, then there is a unique σ-finite measure μ, invariant for P. We then have $m \ll \mu$, and also $\mu(A) > 0 \Rightarrow \sum_{n=0}^{\infty} P^n 1_A(x) = \infty$ for *every* x.

The purpose of this note is to prove the following almost everywhere convergence theorems for functions in $L_p(\mu)$. The results are known for *bounded* functions in $L_p(\mu)$ (see, for example, [2], [8], [5]). Harris-recurrence is treated in [2] and [8].

THEOREM 1. *Let P be Harris-recurrent with finite invariant measure μ. If P is aperiodic, then for every $f \in L_1(\mu)$ we have $P^n f(x) \to \int f d\mu$* a.e.

THEOREM 2. *Let P be Harris recurrent with infinite σ-finite invariant measure μ. Then for every $f \in L_p(\mu)$, $1 \leqq p < \infty$, we have $P^n f(x) \to 0$* a.e.

For the proof we need the following lemma, due to Orey [7] (see also [8]).

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LEMMA 1. *Let Σ be separable and P Harris-recurrent. Then there exist an integer k, a $\Sigma \times \Sigma$ measurable function $q(x,y) \geqq 0$, and sets $B, C \in \Sigma$ (with $\mu(B)\mu(C) > 0$), such that*:

(a) $P^k \geqq Q > 0$, *where* $Qf(x) = \int q(x,y)f(y)\mu(dy)$.

(b) $\inf\{q(x,y): x \in B,\ y \in C\} = \alpha > 0$.

(c) P^k *is Harris-recurrent.*

As a consequence of this lemma, we get that $P^k \geqq Q \geqq \alpha 1_B \otimes \mu I_C$.

The following lemmas are well-known. The case $U = PI_A$ is shown in [2], but the proof is valid for the general case.

LEMMA 2. *If $P > U > 0$, then* $U^n 1 \downarrow 0$ a.e.

PROOF. $U^n 1$ decreases, so let $U^n 1 \downarrow h$. Then $Uh = h \Rightarrow Ph \geqq h \Rightarrow Ph = h$, so $h =$ const, since P is conservative and ergodic in $L_\infty(\mu)$. Hence $Uh = h = c1$, and, if $c \neq 0$, $U1 = 1$. Since $P > U$, we have a contradiction. Hence $c = 0$.

LEMMA 3. *If $P > U > 0$, then* $\Sigma_{n=0}^\infty (P-U)^n U1 = 1$ a.e.

PROOF. Since $U = P - (P - U)$, we have

$$\sum_{n=0}^{N} (P-U)^n U1 = \sum_{n=0}^{N} (P-U)^n 1 - \sum_{n=1}^{N+1} (P-U)^n 1 = 1 - (P-U)^{N+1} 1.$$

Let $N \to \infty$ and apply Lemma 2 (to $P - U$).

LEMMA 4. *Let $P > U > 0$. If $0 \leqq f \in L_p(\mu)$, $1 \leqq p < \infty$ (and $\mu P = \mu$), then* $(P-U)^n f \to 0$ a.e.

PROOF. The proof of Lemma 2 shows that $P - U$ has no superinvariant functions in L_∞. Hence $P - U$ is either dissipative or conservative. Since $\mu(P-U) \leqq \mu$, the same applies to $(P-U)g$ as an operator of $L_1(\mu)$ ([2, p. 76]), and by Lemma 3 we must have that $P - U$ is dissipative. Hence, for $0 \leqq f \in L_1(\mu)$ we have $\Sigma_{n=0}^\infty (P-U)^n f < \infty$ a.e., hence $(P-U)^n f \to 0$ a.e. If $0 \leqq f \in L_p(\mu)$, then $f \leqq 1 + (f-1)^+$, with $(f-1)^+ \in L_1(\mu)$. Hence

$$(P-U)^n f \leqq (P-U)^n 1 + (P-U)^n (f-1)^+ \to 0 \quad \text{a.e.}$$

LEMMA 5. *Let P be Harris-aperiodic with finite invariant measure μ, and let Σ be separable. If $C \in \Sigma$, and $f \in L_1(\mu)$ satisfies $\int f d\mu = 0$, then $\int P^n f \cdot 1_C d\mu \to 0$.*

PROOF. The Harris-recurrence condition implies, by Lemma 1, the Harris condition as given in [2], and the dual Markov operator P^* is also aperiodic Harris [2], and $P^{*n} 1_C \to \mu(C)$ a.e. ([2], [5]). Hence

$$\int P^n f \cdot 1_C d\mu = \int f P^{*n} 1_C d\mu \to \mu(C) \int f d\mu = 0.$$

LEMMA 6. *Let P be Harris with infinite σ-finite invariant measure μ, and let Σ be separable. If $C \in \Sigma$ with $\mu(C) < \infty$, and $0 \leqq f \in L_p(\mu)$, $1 \leqq p < \infty$, then $\int P^n f \cdot 1_C d\mu \to 0$.*

PROOF. If $f \in L_1(\mu)$, the proof is as before, except that $P^{*n} 1_C \to 0$ a.e. If $f \in L_p(\mu)$, $1 < p < \infty$, then for $\varepsilon > 0$ write $f = g + h$, with $g \in L_1(\mu)$ and $\|h\|_p < \varepsilon$. Then $\int P^n g \cdot 1_C d\mu \to 0$, and $0 \leqq \int P^n h \cdot 1_C d\mu \leqq \varepsilon \mu(C)^{1/q}$ (with $q = p/(p-1)$), since P is a contraction of $L_p(\mu)$. The lemma now follows.

The next lemma is an abstraction of the "first entrance formula", and can be proved by induction.

LEMMA 7. *Let a, b be elements of a ring. Then*

$$a^n = \sum_{k=0}^{n-1} (a-b)^k b a^{n-1-k} + (a-b)^n.$$

PROOF OF THEOREMS 1 AND 2. We first assume that Σ is separable, and Lemma 1 can be applied. We may assume that in Lemma 1, $k = 1$ (otherwise we prove the theorem for P^k, and then apply it to the functions $P^j f$, $0 \leqq j \leqq k-1$).

Choose in Lemma 7, $a = P$, $b = \alpha 1_B \otimes \mu I_C$, and obtain

$$P^n f(x) = \sum_{k=0}^{n-1} (P - \alpha 1_B \otimes \mu I_C)^k \alpha 1_B(x) \int P^{n-1-k} f \cdot 1_C d\mu + (P - \alpha 1_B \otimes \mu I_C)^n f(x).$$

For Theorem 1 we assume, w.l.g., $\int f d\mu = 0$. By Lemma 4 the last term tends to 0. Since $\mu(C) < \infty$, by Lemmas 5 and 6 $\int P^n f \cdot 1_C d\mu \to 0$, and, by Lemma 3,

$$\mu(C) \sum_{k=0}^{\infty} (P - \alpha 1_B \otimes \mu I_C)^k \alpha 1_B(x) = 1.$$

After excluding a set of measure 0, for $\varepsilon > 0$ and $x \in X$ there is an N such that

$$\sum_{k=N}^{\infty} (P - \alpha 1_B \otimes \mu I_C)^k \alpha 1_B(x) < \varepsilon,$$

and

$$\int P^n f \cdot 1_C d\mu < \varepsilon \qquad \text{for } n > N.$$

Then, for $n > 2N$, we have

$$\sum_{k=0}^{N} (P - \alpha 1_B \otimes \mu I_C)^k \alpha 1_B(x) \int P^{n-1-k} f \cdot 1_C d\mu < \varepsilon \sum_{k=0}^{N} (P - \alpha 1_B \otimes \mu I_C)^k \alpha 1_B(x) \leqq \varepsilon / \mu(C),$$

and

$$\sum_{k=N+1}^{n-1} (P - \alpha 1_B \otimes \mu I_C)^k \alpha 1_B(x) \int P^{n-1-k} f \cdot 1_C d\mu \leqq \varepsilon \|f\|_p \|1_C\|_q,$$

and the theorems are proved, in case that Σ is separable.

For the case that Σ is not separable, we take the *admissible* σ-field, $\Sigma' \subset \Sigma$ such that f is Σ' measurable, Σ' is separable, and $B(X, \Sigma')$ is invariant for P (see [1, p. 209]).

REMARKS. (1) If we deal with the abstract Harris condition, as used in [2], we can get to the Harris-recurrence condition via the method in [3], or (after reduction to the separable case, as above) via [4].

(2) Theorem 2 is given a different proof in [6, theor. 3]. The assumption there for our Theorem 2 is also of aperiodicity (with only a sketch of proof). The approach in this note is simpler.

REFERENCES

1. J. L. Doob, *Stochastic Processes*, Wiley, New York, 1953.

2. S. R. Foguel, *Ergodic Theory of Markov Processes*, Van Nostrand–Reinhold, New York, 1969.

3. S. Horowitz, *Transition probabilities and contractions of L_∞*, Z. Wahrscheinlichkeitstheorie **24** (1972), 263–274.

4. N. C. Jain, *A note on invariant measures*, Ann. Math. Statist. **37** (1966), 729–732.

5. N. C. Jain, *Some limit theorems for a general Markov process*, Z. Wahrscheinlichkeitstheorie **6** (1966), 206–223.

6. E. Nummelin and R. L. Tweedie, *Geometric ergodicity and R-positivity for general Markov chains*, Ann. Probability, to appear.

7. S. Orey, *Recurrent Markov chains*, Pacific J. Math. **9** (1959), 805–827.

8. S. Orey, *Lecture Notes on Limit Theorems for Markov Chain Transition Probabilities*, Van Nostrand, New York, 1971.

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